What are bounding chains, and why are they related to linking numbers?

Xujia Chen

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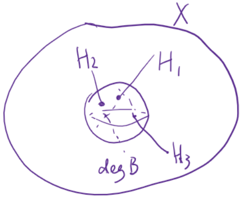
Gromov-Witten invariants

(X, ω): compact symplectic manifold of dim $2n$
J: generic ω-tame almost complex structure
$B \in H_2(X)$,
$H_1, \ldots, H_l \subset X$: closed submanifolds in general position
such that $2(c_1(TX) \cdot B + n - 3 + l) = \sum_i \mathrm{codim} H_i$,
$[H_1], \ldots, [H_l] \in H_*(X)$: their homology classes,

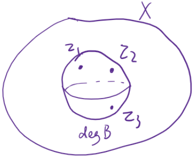
Gromov-Witten invariants
$\langle [H_1], \ldots, [H_l] \rangle_B^X \equiv$
Number of degree B J-holomorphic rational curves in X passing through $H_1, \ldots, H_l$



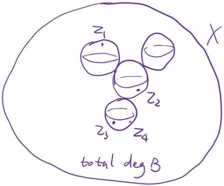
This number does not depend on the choices of J and $H_1, \ldots, H_l$ in $[H_1], \ldots, [H_l]$.

Gromov-Witten invariants

$\mathfrak{M}_l(B) \equiv$ moduli space of deg B rational J-holomorphic curves in X with l marked points (can be viewed as a smooth manifold)



$\overline{\mathfrak{M}}_l(B) \equiv$ moduli space of deg B *nodal* rational J-holomorphic curves in X with l marked points (can be viewed as a compact, smooth manifold)



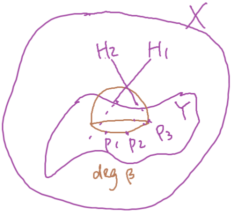
$\langle [H_1], \ldots, [H_l] \rangle_B^X =$ intersection number of

$$\overline{\mathfrak{M}}_l(B) \xrightarrow{\text{ev}} \underbrace{X \times \ldots \times X}_{l \text{ times}} \hookleftarrow H_1 \times \ldots \times H_l$$

open Gromov-Witten "invariants"

$(X,\omega), J$ as before
$Y \subset X$ Lagrangian submanifold, oriented, Spin
$\beta \in H_2(X,Y)$,
$H_1,\dots,H_l \subset X$: even-dimensional closed submanifolds,
$p_1,\dots,p_k \in Y$ points
such that $\mu(\beta)+n-3+k+2l = \sum_i \mathrm{codim} H_i + kn$,

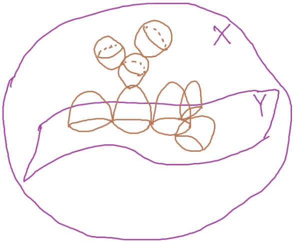
Define
$\mathrm{Disk}(\beta, \{p_i\}_{i=1}^k, \{H_i\}_{i=1}^l) \equiv$
$\{$degree β J-holomorphic disks in X
with boundaries in Y,
passing through $p_1,\dots,p_k,H_1,\dots,H_l.\}$



open Gromov-Witten "invariants"

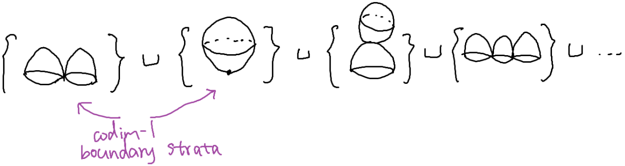
$\mathfrak{M}_{k,l}(\beta) \equiv$ moduli space of deg β J-holomorphic disks in X whose boundary lies in Y, with k boundary and l interior marked points (can be viewed as a smooth manifold)

$\overline{\mathfrak{M}}_{k,l}(\beta) \equiv$ moduli space of deg β *nodal* J-hol disks in X whose boundary lies in Y, with k boundary and l interior marked points (can be viewed as a compact, smooth manifold *with boundary*)



open Gromov-Witten "invariants"

$\overline{\mathfrak{M}}_{k,l}(\beta) - \mathfrak{M}_{k,l}(\beta) =$



$\left|\mathsf{Disk}(\beta, \{p_i\}, \{H_i\})\right|^{\pm} =$ intersection number of

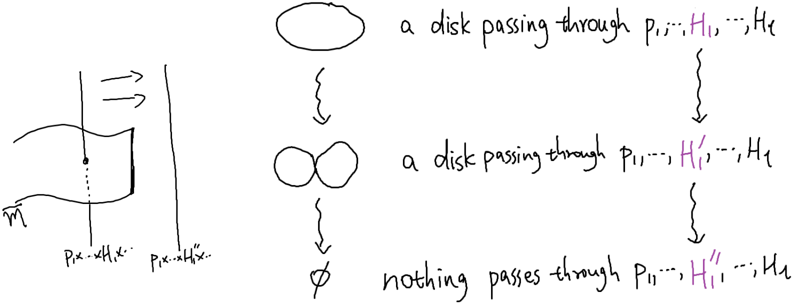
$$\overline{\mathfrak{M}}_{k,l}(\beta) \xrightarrow{\mathsf{ev}} \underbrace{Y \times \ldots \times Y}_{k \text{ times}} \times \underbrace{X \times \ldots \times X}_{l \text{ times}} \hookleftarrow p_1 \times \ldots \times p_k \times H_1 \times \ldots \times H_l$$

Not an invariant: it may depend on $J, p_1, \ldots, p_k, H_1, \ldots, H_l$.

open Gromov-Witten "invariants"

$$\overline{\mathfrak{M}}_{k,l}(\beta) \xrightarrow{\text{ev}} \underbrace{Y \times \ldots \times Y}_{k \text{ times}} \times \underbrace{X \times \ldots \times X}_{l \text{ times}} \hookleftarrow p_1 \times \ldots \times p_k \times H_1 \times \ldots \times H_l$$

e.g. we move H_1 around:



Q:How do we define invariant disk counts?

Some definitions relevant to today

Fukaya(2011) defined open GW-invariants for Calabi-Yau 3-folds in terms of A_∞-algebras of differential forms

Welschinger(2013) defined invariant disk counts when $n = 3$, which counts multi-disks with "self-linking numbers"

(Tian expressed belief that Welschinger's definition is a geometric interpretation of Fukaya-type algebra.)

Solomon-Tukachinsky(2016) defined open GW-invariants generalizing Fukaya(2011) for all n odd, if Y is a $\mathbb{Q}$-homology sphere. Their construction is based on the idea of "bounding chains" in FOOO(2006).

My work in 2019 translated Solomon-Tukachinsky's construction into a more geometric language, from which it immediate follows that these invariants are the same as Welschinger's when $n = 3$.

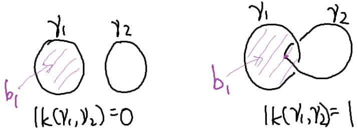
Today: show you what this construction looks like.

Welschinger's definition

Welschinger(2013) defined invariant disk counts when $n = 3$ and $H_1(Y) \hookrightarrow H_1(X)$.
Idea: count multi-disks weighted by "self-linking number".

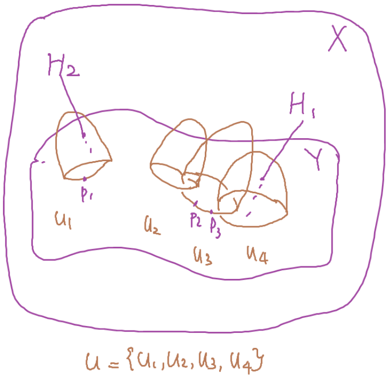
Recall:

dim X=6, dim Y=3
For γ_1, γ_2 oriented loops in Y, such that $\gamma_1 \cap \gamma_2 = \emptyset, \gamma_1 = \partial b_1, \gamma_2 = \partial b_2$, their linking number is defined as lk$(\gamma_1, \gamma_2) \equiv |\gamma_1 \cap b_2|^{\pm}$.

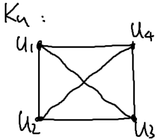


Welschinger's definition

Idea: count multi-disks weighted by "self-linking number".



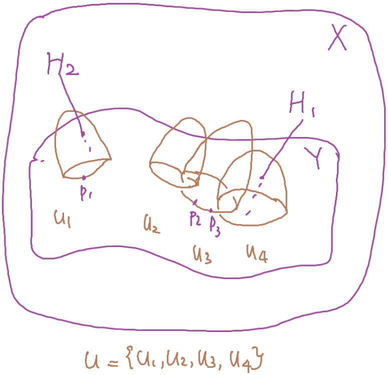
For u a multi-disk as shown, define
$K_u =$ complete graph with vertex set $\{u_i\}$,



$$\mathsf{lk}(u) \equiv \sum_{\substack{T:\text{ spanning tree}\\ \text{in } K_u}} \prod_{\substack{e:\text{ an edge}\\ \text{connecting } u_i, u_j}} \mathsf{lk}(\partial u_i, \partial u_j)$$

Welschinger's definition

Idea: count multi-disks weighted by "self-linking number".



We count u with weight $\pm \mathrm{lk}(u)$.

The count is independent of the choices of $J, \{p_i\}$ and $H_i \in [H_i]$.

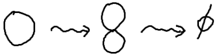
Welschinger's definition

Idea: count multi-disks weighted by "self-linking number".

Why invariant?

Let us move, say, H_1 around:

$$H_1 \rightsquigarrow H_1' \rightsquigarrow H_1''$$



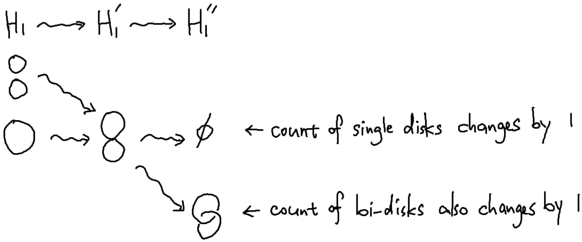
← count of single disks changes by 1

Welschinger's definition

Idea: count multi-disks weighted by "self-linking number".

Why invariant?

Let us move, say, H_1 around:



Attempt of generalizing to higher dimensions

Recall: for $\beta \in H_2(X,Y), K \subset \{p_1,\dots,p_k\}, L \subset \{H_1,\dots,H_l\}$,

$$\dim \text{Disk}(\beta,K,L) = \mu(\beta)+n-3-|K|(n-1)-\sum_{H_i\in L}(\text{codim } H_i - 2)$$

Suppose $\big(\beta,\{p_1,\dots,p_k\},\{H_1,\dots,H_l\}\big)$ is such that
$\dim \text{Disk}(\beta,\{p_i\},\{H_i\}) = 0$, and
$\beta=\beta_1+\beta_2, \{p_1,\dots,p_k\}=K_1\sqcup K_2, \{H_1,\dots,H_l\}=L_1\sqcup L_2$, then

$$\dim \text{Disk}(\beta_1,K_1,L_1)+\dim \text{Disk}(\beta_2,K_2,L_2)=n-3.$$

Both terms can be positive, so "counting bi-disks" no longer makes sense.

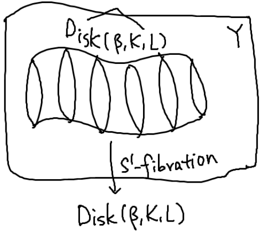
Attempt of generalizing to higher dimensions

Suppose $(\beta, \{p_1, \ldots, p_k\}, \{H_1, \ldots, H_l\})$ is such that dim Disk$(\beta, \{p_i\}, \{H_i\})$ =0, and $\beta = \beta_1 + \beta_2, \{p_1, \ldots, p_k\} = K_1 \sqcup K_2, \{H_1, \ldots, H_l\} = L_1 \sqcup L_2$, then

$$\text{dim Disk}(\beta_1, K_1, L_1) + \text{dim Disk}(\beta_2, K_2, L_2) = n - 3.$$

Denote

$$\text{Disk}\widehat{(\beta, K}, L) = \bigsqcup_{u \in \text{Disk}(\beta, K, L)} \partial u,$$



then

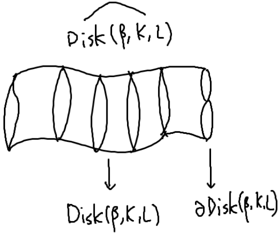
$$\text{dim Disk}(\widehat{\beta_1, K_1}, L_1) + \text{dim Disk}(\widehat{\beta_2, K_2}, L_2) = n - 1,$$

the correct dimension to define linking numbers in Y.

Attempt of generalizing to higher dimensions

So, we can define "count of $(\beta_1, K_1, L_1), (\beta_2, K_2, L_2)$-bi-disks" to be the linking number between Disk$\widehat{(\beta_1, K_1}, L_1)$, Disk$\widehat{(\beta_2, K_2}, L_2)$, if they are closed.

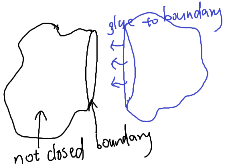
Not the case: Disk$\widehat{(\beta, K}, L)$ usually has boundary



Goal: for every triple (β, K, L), we want to make Disk$\widehat{(\beta, K}, L)$ closed.

The idea of bounding chains

We are going to "close up" $\mathrm{Disk}(\widehat{\beta, K}, L)$ inductively, by gluing manifolds to its boundaries.



Suppose for every triple (β', K', L') such that

$$\beta' < \beta, K' \subset K, L' \subset L,$$

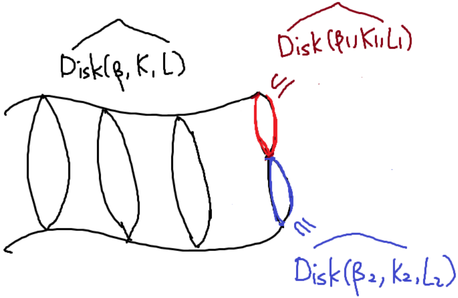
we have already constructed a closed submanifold $\mathfrak{bb}(\beta', K', L')$ of Y containing $\mathrm{Disk}(\widehat{\beta', K'}, L')$.

Since Y is a homology sphere, we can take $\mathfrak{b}(\beta', K', L')$ a submanifold with boundary in Y, such that

$$\partial \mathfrak{b}(\beta', K', L') = \mathfrak{bb}(\beta', K', L')$$

The idea of bounding chains

Then, for a boundary component of $\widehat{\text{Disk}(\beta, K, L)}$:



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The idea of bounding chains

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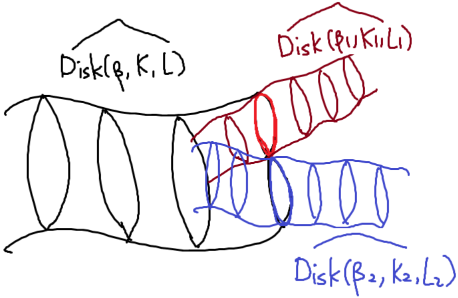

=boundaries of disks in Disk(β_1, K_1, L_1) passing through Disk$(\widehat{\beta_2, K_2}, L_2)$

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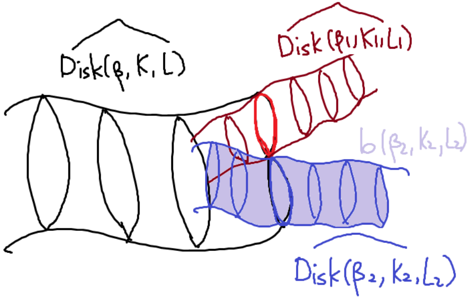
The idea of bounding chains

Then, for a boundary component of $\widehat{\mathrm{Disk}(\beta, K, L)}$:



 =boundaries of disks in $\mathrm{Disk}(\beta_1, K_1, L_1)$ passing through $\widehat{\mathrm{Disk}(\beta_2, K_2, L_2)}$

We consider the boundaries of disks in $\mathrm{Disk}(\beta_1, K_1, L_1)$ passing through $\mathfrak{b}(\beta_2, K_2, L_2)$, and glue this space to .

This closes up the red part of the boundary of $\widehat{\mathrm{Disk}(\beta, K, L)}$.

The idea of bounding chains

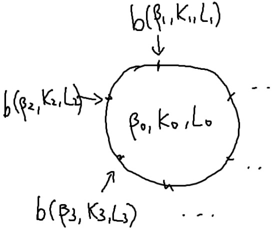
More precisely...

We define

$$\mathfrak{bb}(\beta, K, L) = \bigsqcup_{\eta} \overbrace{\mathsf{Disk}\Big(\beta_0, K_0 \sqcup \big\{\mathfrak{b}(\beta_i, K_i, L_i)\big\}_{i=1}^{k}, L_0\Big)},$$

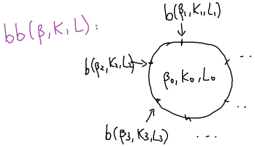
where η stands for all possible ways to write

$$\beta = \beta_0 + \beta_1 + \ldots + \beta_k$$
$$K = K_0 \sqcup K_1 \sqcup K_2 \sqcup \ldots \sqcup K_k$$
$$L = L_0 \sqcup L_1 \sqcup L_2 \sqcup \ldots \sqcup L_k$$



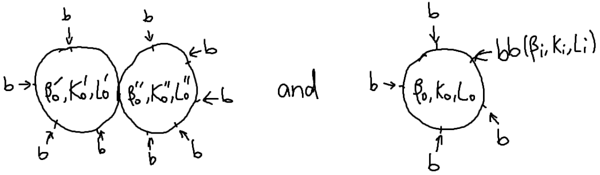
i.e. all possible ways to split (β, K, L) into a "base" (β_0, K_0, L_0) and "branches" $(\beta_i, K_i, L_i), 1 \leq i \leq k$.

The idea of bounding chains

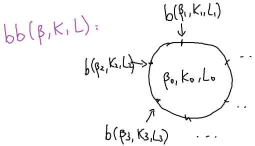


Why is $\mathfrak{bb}(\beta, K, L)$ closed?

$\partial\mathfrak{bb}(\beta, K, L)$ are of two kinds:



The idea of bounding chains



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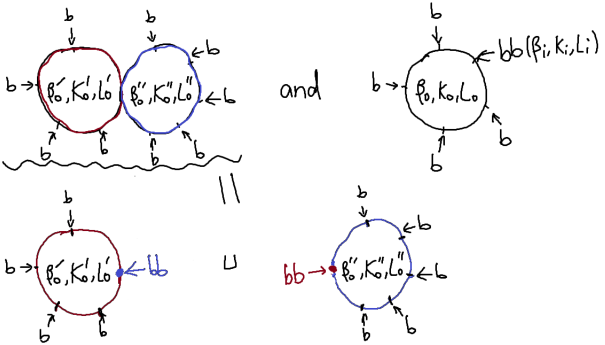
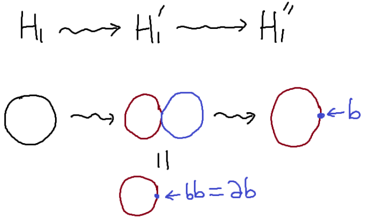


Because we are summing over all possible splits of (β, K, L), they cancel with each other!

The idea of bounding chains

Now we have defined $\mathfrak{bb}(\beta, K, L)$ for each triple (β, K, L), we can define open GW-invariants
"counts of degree β disks passing through $\{p_0\} \sqcup K$ and L"
to be $\left|\mathfrak{bb}(\beta, K, L) \cap \{p_0\}\right|^{\pm}$.

Morally, why invariant:



Thank you!