Lifting cobordisms and Kontsevich-type recursions for counts of real curves

Xujia Chen

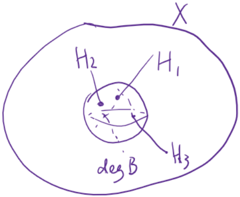
Gromov-Witten invariants

(X, ω): compact symplectic manifold of dim $2n$
J: generic ω-tame almost complex structure
$B \in H_2(X)$,
$H_1, \dots, H_l \subset X$: closed submanifolds in general position
such that $2(c_1(TX)\cdot B+n-3+l) = \sum_i \mathrm{codim} H_i$,
$[H_1], \dots, [H_l] \in H_*(X)$: their homology classes,

Gromov-Witten invariants

(X, ω): compact symplectic manifold of dim $2n$
J: generic ω-tame almost complex structure
$B \in H_2(X)$,
$H_1, \dots, H_l \subset X$: closed submanifolds in general position such that $2(c_1(TX)\cdot B + n - 3 + l) = \sum_i \mathrm{codim} H_i$,
$[H_1], \dots, [H_l] \in H_*(X)$: their homology classes,

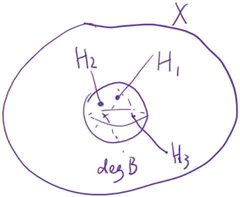
Gromov-Witten invariants
$\langle [H_1], \dots, [H_l] \rangle_B^X \equiv$
Number of degree B J-holomorphic rational curves in X passing through $H_1, \dots, H_l$



Gromov-Witten invariants

(X, ω): compact symplectic manifold of dim $2n$
J: generic ω-tame almost complex structure
$B \in H_2(X)$,
$H_1, \dots, H_l \subset X$: closed submanifolds in general position such that $2(c_1(TX) \cdot B + n - 3 + l) = \sum_i \mathrm{codim} H_i$,
$[H_1], \dots, [H_l] \in H_*(X)$: their homology classes,

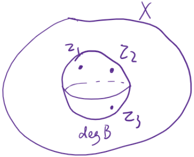
Gromov-Witten invariants
$\langle [H_1], \dots, [H_l] \rangle_B^X \equiv$
Number of degree B J-holomorphic rational curves in X passing through $H_1, \dots, H_l$



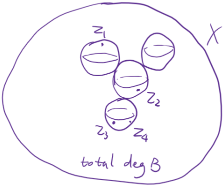
This number does not depend on the choices of J and $H_1, \dots, H_l$ in $[H_1], \dots, [H_l]$.

Gromov-Witten invariants

$\mathfrak{M}_l(B) \equiv$ moduli space of deg B rational J-holomorphic curves in X with l marked points (can be viewed as a smooth manifold)



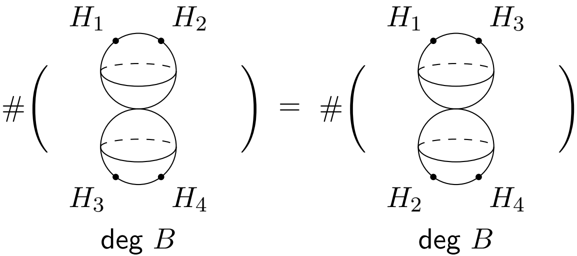
$\overline{\mathfrak{M}}_l(B) \equiv$ moduli space of deg B *nodal* rational J-holomorphic curves in X with l marked points (can be viewed as a compact, smooth manifold)



$$\langle [H_1], \ldots, [H_l] \rangle_B^X = \text{intersection number of}$$

$$\overline{\mathfrak{M}}_l(B) \xrightarrow{\text{ev}} \underbrace{X \times \ldots \times X}_{l \text{ times}} \hookleftarrow H_1 \times \ldots \times H_l$$

WDVV Relations (Kontsevich '92, Ruan-Tian '93)



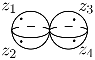
(both sides also pass through $H_5, \dots, H_l$)

Together with *splitting formulas* (expressing a nodal count as counts of its components), this gives relations for Gromov-Witten invariants.

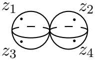
Idea of proof — lifting homology relation from $\overline{\mathcal{M}}_{0,4}$

$$\overline{\mathcal{M}}_{0,4} \equiv \{(z_1, z_2, z_3, z_4) : z_i \in \mathbb{P}^1\}/\mathsf{Aut}(\mathbb{P}^1) \quad \approx \mathbb{P}^1$$

$$[(z_1, z_2, z_3, z_4)] \quad \to \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$



$$= \sigma_0 \quad \to 0$$

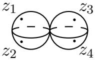


$$= \sigma_\infty \quad \to \infty$$

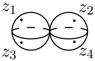
Idea of proof — lifting homology relation from $\overline{\mathcal{M}}_{0,4}$

$$\overline{\mathcal{M}}_{0,4} \equiv \{(z_1, z_2, z_3, z_4) : z_i \in \mathbb{P}^1\}/\mathsf{Aut}(\mathbb{P}^1) \quad \approx \mathbb{P}^1$$

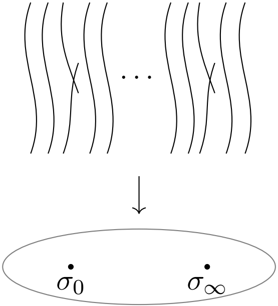
$$[(z_1, z_2, z_3, z_4)] \quad \to \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$



$$= \sigma_0 \quad \to 0$$



$$= \sigma_\infty \quad \to \infty$$



$\mathcal{U}\overline{\mathcal{M}}_{0,4}$: universal family over $\overline{\mathcal{M}}_{0,4}$

$$\pi \downarrow$$

$$\overline{\mathcal{M}}_{0,4}$$

Idea of proof — lifting homology relation from $\overline{\mathcal{M}}_{0,4}$

$$\overline{\mathfrak{M}}_l(B) \xrightarrow{\text{ev}} X^l \hookleftarrow H_1 \times \ldots \times H_l$$

$$\Big\downarrow \text{f: forgetful morphism, mapping a curve to its first 4 marked points}$$

$$\overline{\mathcal{M}}_{0,4}$$

$$[\sigma_0] = [\sigma_\infty] \in H_0(\overline{\mathcal{M}}_{0,4}) \Longrightarrow [f^{-1}(\sigma_0)] = [f^{-1}(\sigma_\infty)] \in H_*\big(\overline{\mathfrak{M}}_l(B)\big)$$

$$\Longrightarrow \text{The intersection numbers} \begin{cases} f^{-1}(0) \xrightarrow{\text{ev}} X^l \hookleftarrow H_1 \times \ldots \times H_l \\ f^{-1}(\infty) \xrightarrow{\text{ev}} X^l \hookleftarrow H_1 \times \ldots \times H_l \end{cases}$$

are equal.

Real case

(X, ω, ϕ): compact real symplectic manifold of dim $2n$
(real means $\phi : X \to X, \phi^*\omega = -\omega, \phi^2 = \mathrm{id}$),
J: real (i.e. $\phi^* J = -J$) ω-tame almost complex structure.
$X^\phi :=$ fixed locus of ϕ. $B \in H_2(X)$.
A rational curve $C \subset X$ is called real if $\phi(C) = C$.

e.g. $X = \mathbb{CP}^n, \phi([z_0, \dots, z_n]) = [\bar{z}_0, \dots, \bar{z}_n]$; $X^\phi = \mathbb{RP}^n$.
Every curve given by polynomials with real coefficients is real.

$H_1, \dots, H_l \subset X$ closed submanifolds, $p_1, \dots, p_k \in X^\phi$ points, s.t.
$c_1(X)B + n - 3 + k + 2l = \sum_{i=1}^{l} \mathrm{codim} H_i + nk$.

Q: Counts of real curves?

Welschinger invariants

Suppose $n = 2$ or 3. Suppose X^ϕ is oriented in $n = 3$ case.

Theorem (Welschinger '03,'05)

The number of degree B real rational J-holomorphic curves passing through $H_1, \ldots, H_l, p_1, \ldots, p_k$, counted with appropriate signs, is independent of J, $p_1, \ldots, p_k$ and $H_1, \ldots, H_l$ in $[H_1], \ldots, [H_l] \in H_*(X - X^\phi)$.

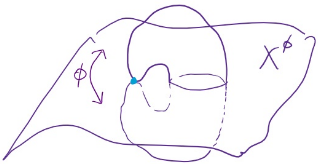
Welschinger invariants

Suppose $n=2$ or 3. Suppose X^ϕ is oriented in $n=3$ case.

Theorem (Welschinger '03,'05)

The number of degree B real rational J-holomorphic curves passing through $H_1,\ldots,H_l,p_1,\ldots,p_k$, counted with appropriate signs, is independent of J, $p_1,\ldots,p_k$ and $H_1,\ldots,H_l$ in $[H_1],\ldots,[H_l]\in H_*(X-X^\phi)$.

In $n=2$ case, a curve C is counted with $(-1)^{\#(\text{isolated real nodes of } C)}$.



Welschinger invariants

These numbers are called *Welschinger invariants* $\langle [H_1], \ldots, [H_l] \rangle_B^{\phi}$.

Welschinger invariants

These numbers are called *Welschinger invariants* $\langle [H_1], \ldots, [H_l] \rangle_B^{\phi}$.

They provide lower bounds of the number of real curves.

Welschinger invariants

These numbers are called *Welschinger invariants* $\langle [H_1], \ldots, [H_l] \rangle_B^\phi$.

They provide lower bounds of the number of real curves.

Example: first few numbers for $\mathbb{CP}^2$ (with $H_1,\ldots,H_l$ being points)

	d=1	d=2	d=3
l=0	1	1	8,10,12
l=1	1	1	6,8,10,12
l=2		1	4,6,8,10,12
l=3			2,4,...,12
l=4			0,2,...,12

Real WDVV

Solomon'07 proposed WDVV-type relations for Welschinger invariants for symplectic 4-folds.

Real WDVV

Solomon'07 proposed WDVV-type relations for Welschinger invariants for symplectic 4-folds.

Theorem (C.'18): *When dim $X = 4$,*

Solomon's relations for Welschinger invariants hold.

Theorem (C.-Zinger'19): *When dim $X = 6$,*

similar formulas hold for (X, ω, ϕ) with some finite symmetry.
e.g. $\mathbb{CP}^3$ with a real hyperplane reflection.

Real WDVV

Solomon'07 proposed WDVV-type relations for Welschinger invariants for symplectic 4-folds.

Theorem (C.'18): *When dim $X = 4$,*

Solomon's relations for Welschinger invariants hold.

Theorem (C.-Zinger'19): *When dim $X = 6$,*

similar formulas hold for (X, ω, ϕ) with some finite symmetry.
e.g. $\mathbb{CP}^3$ with a real hyperplane reflection.

For many symplectic 4-folds and 6-folds, they completely determine all Welschinger invariants recursively.
e.g. $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, real blow-ups of $\mathbb{P}^2$, $\mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Real WDVV

Solomon'07 proposed WDVV-type relations for Welschinger invariants for symplectic 4-folds.

Theorem (C.'18): *When dim X=4,*

Solomon's relations for Welschinger invariants hold.

Theorem (C.-Zinger'19): *When dim X=6,*

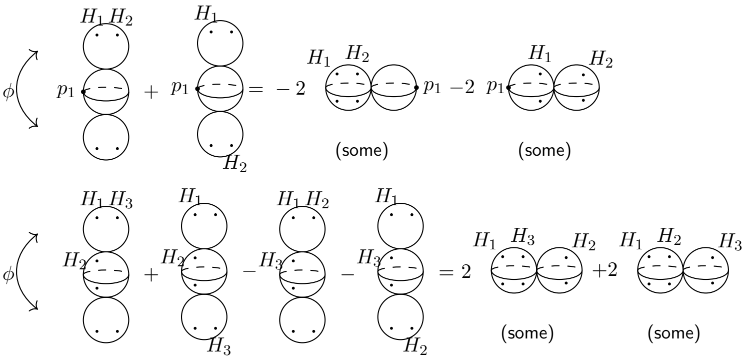
similar formulas hold for (X, ω, ϕ) with some finite symmetry.
e.g. $\mathbb{CP}^3$ with a real hyperplane reflection.

For many symplectic 4-folds and 6-folds, they completely determine all Welschinger invariants recursively.
e.g. $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, real blow-ups of $\mathbb{P}^2$, $\mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Remark: Solomon-Tukachinsky'19 open WDVV-equations cover dimension 6 case above, but not dimension 4, since X^ϕ does not need to be orientable.

Real WDVV

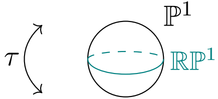
Each diagram represents counts of curves of such shape (of total degree B, and passing through the rest of the constraints as well).



Together with splitting formulas, these two equations give WDVV-type relations for Welschinger invariants.

Sketch of proof

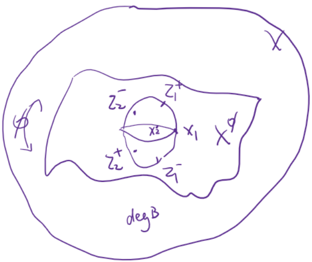
$$\tau : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$
$$[z_0, z_1] \longrightarrow [\bar{z}_0, \bar{z}_1]$$

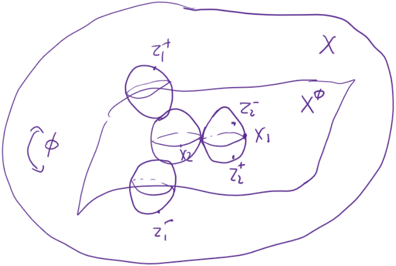


$$\mathbb{R}\mathfrak{M}_{k,l}(B) \equiv \{u : \mathbb{P}^1 \to X, J\text{-holomorphic}, [u] = B, \phi \circ u = u \circ \tau, z_1^{\pm}, \ldots, z_l^{\pm} \in \mathbb{P}^1, z_i^- = \tau(z_i^+), x_1, \ldots, x_k \in \mathbb{RP}^1\} / \mathrm{Aut}_{\mathbb{R}}(\mathbb{P}^1)$$

$=$moduli space of degree B real rational J-holomorphic curves with k real and l conjugate pairs of marked points

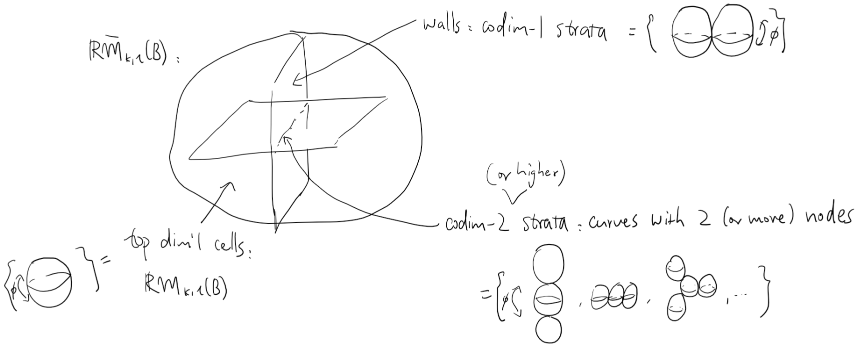
$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \equiv$ its compactification by adding in real nodal curves

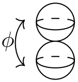




Sketch of proof

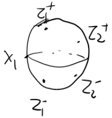
$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$ has a stratified space structure.



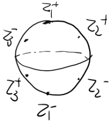
Remark: Curves of shape  never appear in our cases. In dim 4 this follows from the dimension condition; in dim 6 this is taken care of by the finite symmetry condition.

Sketch of proof

$\mathbb{R}\overline{\mathcal{M}}_{1,2} \equiv$ moduli space of the relative position of 1 real and 2 conjugate pairs of marked points on $\mathbb{P}^1$



$\mathbb{R}\overline{\mathcal{M}}_{0,3} \equiv$ moduli space of the relative position of 3 conjugate pairs of marked points on $\mathbb{P}^1$



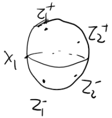
$$\begin{array}{ccc} \mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) & \xrightarrow{\quad \text{ev} \quad} & (X^{\phi})^k \times X^l \\ \Big\downarrow \text{f} & & \\ \mathbb{R}\overline{\mathcal{M}}_{1,2} (\text{resp. } \mathbb{R}\overline{\mathcal{M}}_{0,3}) & & \end{array}$$

$$\text{ev}([u, x_1, \ldots, z_l^{\pm}]) = \big(u(x_1), \ldots, u(x_k), u(z_1^{+}), \ldots, u(z_l^{+})\big)$$

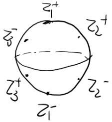
$$\text{f}([u, x_1, \ldots, z_l^{\pm}]) = (x_1, z_1^{\pm}, z_2^{\pm})\big(\text{resp. } (z_1^{\pm}, z_2^{\pm}, z_3^{\pm})\big)$$

Sketch of proof

$\mathbb{R}\overline{\mathcal{M}}_{1,2} \equiv$ moduli space of the relative position of 1 real and 2 conjugate pairs of marked points on $\mathbb{P}^1$



$\mathbb{R}\overline{\mathcal{M}}_{0,3} \equiv$ moduli space of the relative position of 3 conjugate pairs of marked points on $\mathbb{P}^1$



$$\begin{array}{ccc} \mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) & \xrightarrow{\text{ev}} & (X^{\phi})^k \times X^l \\ \big\downarrow \text{f} & & \\ \mathbb{R}\overline{\mathcal{M}}_{1,2}(\text{resp. } \mathbb{R}\overline{\mathcal{M}}_{0,3}) & & \end{array}$$

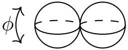
May not be orientable!

$$\text{ev}([u, x_1, \ldots, z_l^{\pm}]) = \big(u(x_1), \ldots, u(x_k), u(z_1^+), \ldots, u(z_l^+)\big)$$
$$\text{f}([u, x_1, \ldots, z_l^{\pm}]) = (x_1, z_1^{\pm}, z_2^{\pm})\big(\text{resp. } (z_1^{\pm}, z_2^{\pm}, z_3^{\pm})\big)$$

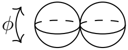
Sketch of proof

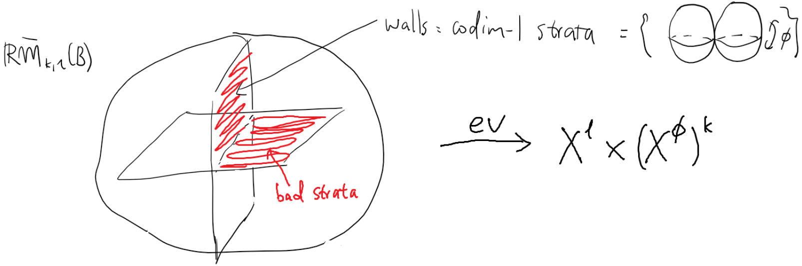
Idea: lifting homology relations from $\mathbb{R}\overline{\mathcal{M}}_{1,2}$ and $\mathbb{R}\overline{\mathcal{M}}_{0,3}$, *together with bounding manifolds*, to incorporate wall-crossing effects coming from walls that obstruct relative-orientability.

Sketch of proof

- Curves of shape  form (real) codim-1 strata in $\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$.
- Solomon'06 $\Longrightarrow$ $\mathrm{ev}|_{\mathbb{R}\mathfrak{M}_{k,l}(B)}$ is relatively orientable, relative orientation extends through some codim-1 strata, but not the others (let's call them "bad strata".)

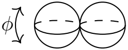
Sketch of proof

- Curves of shape  form (real) codim-1 strata in $\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$.
- Solomon'06 $\Longrightarrow$ $\mathrm{ev}|_{\mathbb{R}\mathfrak{M}_{k,l}(B)}$ is relatively orientable, relative orientation extends through some codim-1 strata, but not the others (let's call them "bad strata".)



ev is relatively orientable out of the red walls.

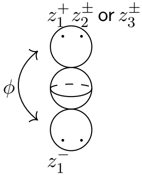
Sketch of proof

- Curves of shape  form (real) codim-1 strata in $\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$.
- Solomon'06 $\Longrightarrow$ $\mathrm{ev}|_{\mathbb{R}\mathfrak{M}_{k,l}(B)}$ is relatively orientable, relative orientation extends through some codim-1 strata, but not the others (let's call them "bad strata".)

($f : X \longrightarrow Y$ is *relatively orientable* if $f^*w_1(Y) = w_1(X)$; equivalently, for every $x \in X$ we have an identification between the orientations of T_xX and $T_{f(x)}Y$, which varies continuously with x.)

Sketch of proof

Let $\Gamma \subset \mathbb{R}\overline{\mathcal{M}}_{1,2}$ (resp. $\mathbb{R}\overline{\mathcal{M}}_{0,3}$) consist of curves of shape

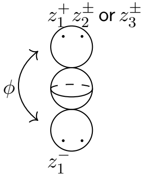


.

Georgieva-Zinger'13 $\Longrightarrow$ Γ is a codim-2 submanifold that bounds in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$.

Sketch of proof

Let $\Gamma \subset \mathbb{R}\overline{\mathcal{M}}_{1,2}$ (resp. $\mathbb{R}\overline{\mathcal{M}}_{0,3}$) consist of curves of shape



Georgieva-Zinger'13 $\Longrightarrow$ Γ is a codim-2 submanifold that bounds in $\mathbb{R}\overline{\mathcal{M}}_{0,3}$.

We take $Y \subset \mathbb{R}\overline{\mathcal{M}}_{1,2}$ (resp. $\mathbb{R}\overline{\mathcal{M}}_{0,3}$) s.t.

- $\partial Y = \Gamma$, and
- $Y \hookrightarrow \mathbb{R}\overline{\mathcal{M}}_{1,2}$ (resp. $\mathbb{R}\overline{\mathcal{M}}_{0,3}$) is relatively oriented, i.e. $\mathcal{N}Y$ is oriented.

Sketch of proof

Then, in

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \xrightarrow{\text{ev}\times\text{f}} (X^{\phi})^k \times X^l \times \mathbb{R}\overline{\mathcal{M}}_{\substack{1,2\\(0,3)}} \hookleftarrow \underbrace{(p_1 \times \ldots \times p_k \times H_1 \times \ldots \times H_l)}_{\text{denote by } C} \times Y,$$

Sketch of proof

Then, in

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \xrightarrow{\mathsf{ev}\times\mathsf{f}} (X^{\phi})^k \times X^l \times \mathbb{R}\overline{\mathcal{M}}_{\substack{1,2\\(0,3)}} \hookleftarrow \underbrace{(p_1 \times \ldots \times p_k \times H_1 \times \ldots \times H_l)}_{\text{denote by } C} \times Y,$$

the intersection numbers

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \cdot (C \times \Gamma) = \pm 2 \text{ ``bad strata''} \cdot (C \times Y). \qquad (\star)$$

Sketch of proof

Then, in

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \xrightarrow{\text{ev}\times\text{f}} (X^{\phi})^k \times X^l \times \mathbb{R}\overline{\mathcal{M}}_{\substack{1,2\\(0,3)}} \hookleftarrow \underbrace{(p_1 \times \ldots \times p_k \times H_1 \times \ldots \times H_l)}_{\text{denote by } C} \times Y,$$

the intersection numbers

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \cdot (C \times \Gamma) = \pm 2 \text{ “bad strata”} \cdot (C \times Y). \qquad (\star)$$

- LHS of $(\star)$ would be 0 if everything is orientable.
- RHS of $(\star)$ is the wall-crossing effect from obstructions to relative orientability.

Sketch of proof

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \cdot (C \times \Gamma) = \pm 2 \text{ “bad strata”} \cdot (C \times Y). \qquad (\star)$$

Proof of $(\star)$:

$\mathbb{R}\widehat{\mathfrak{M}}(B) \equiv$ cut $\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$ open along bad strata

— so it becomes a manifold with boundary. Then, in

$$\mathbb{R}\widehat{\mathfrak{M}}(B) \xrightarrow{\text{ev}\times\text{f}} (X^{\phi})^k \times X^l \times \mathbb{R}\overline{\mathcal{M}}_{1,2} \hookleftarrow C \times Y,$$

Sketch of proof

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B) \cdot (C \times \Gamma) = \pm 2 \text{ “bad strata”} \cdot (C \times Y). \qquad (\star)$$

Proof of $(\star)$:

$\mathbb{R}\widehat{\mathfrak{M}}(B) \equiv$ cut $\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$ open along bad strata

— so it becomes a manifold with boundary. Then, in

$$\mathbb{R}\widehat{\mathfrak{M}}(B) \xrightarrow{\text{ev}\times\text{f}} (X^\phi)^k \times X^l \times \mathbb{R}\overline{\mathcal{M}}_{1,2} \hookleftarrow C \times Y,$$

relatively orientable

Sketch of proof

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)\cdot(C\times\Gamma)=\pm 2\text{ “bad strata”}\cdot(C\times Y). \qquad (\star)$$

Proof of $(\star)$:

$\mathbb{R}\widehat{\mathfrak{M}}(B)\equiv$ cut $\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)$ open along bad strata

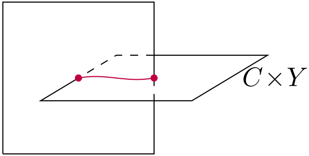
— so it becomes a manifold with boundary. Then, in

$$\boxed{\mathbb{R}\widehat{\mathfrak{M}}(B)\xrightarrow{\text{ev}\times\text{f}}(X^{\phi})^k\times X^l}\times\boxed{\mathbb{R}\overline{\mathcal{M}}_{1,2}\hookleftarrow C\times Y},$$

relatively orientable

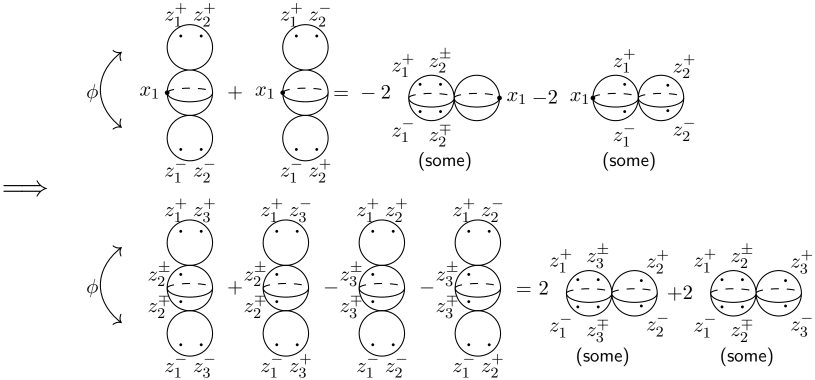
$$0=\partial\big((\mathbb{R}\widehat{\mathfrak{M}}\cdot(C\times Y)\big)=\mathbb{R}\widehat{\mathfrak{M}}\cdot\underbrace{\partial(C\times Y)}_{C\times\Gamma}\pm\underbrace{(\partial\mathbb{R}\widehat{\mathfrak{M}})}_{2\text{ “bad strata”}}\cdot(C\times Y).$$

$\mathbb{R}\widehat{\mathfrak{M}}(B)$



Sketch of proof

$$\mathbb{R}\overline{\mathfrak{M}}_{k,l}(B)\cdot(C\times\Gamma)=\pm 2\text{ "bad strata"}\cdot(C\times Y).\qquad (\star)$$



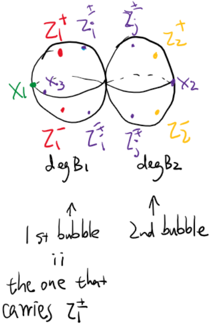
LHSs = counts of curves represented by $\Gamma\subset\mathbb{R}\overline{\mathcal{M}}_{1,2}$ (resp. $\mathbb{R}\overline{\mathcal{M}}_{0,3}$)

RHSs = counts of curves in "bad strata", cut out by Y

Sketch of proof – splitting

Splitting of RHS of $(\star)$:

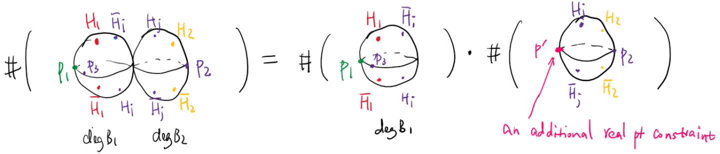
a dimension count + good choice of $Y \implies$



For all bad strata contributing to RHS,

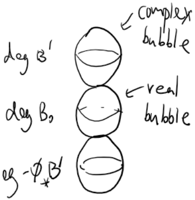
- 1st bubble is rigid
- cut out by Y = fixing position of node on the 1st bubble

$$\Downarrow$$



Sketch of proof – splitting

Splitting of LHS of $(\star)$:



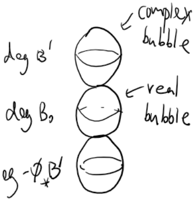
n=2: immediate, since generically two curves intersect at a fixed number of points in X

n=3: a dimension count $\Rightarrow$ only two cases:

- real bubble is rigid $\Rightarrow$ #(nodal curves)= #(real bubble)·#(complex bubble, with an additional constraint)
- complex bubble is rigid $\Rightarrow$ #(nodal curves)= #(complex bubble)·#(real bubble, with an additional complex constraint)

Sketch of proof – splitting

Splitting of LHS of ($\star$):



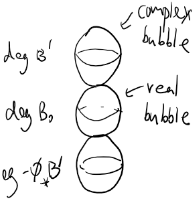
n=2: immediate, since generically two curves intersect at a fixed number of points in X

n=3: a dimension count $\Rightarrow$ only two cases:

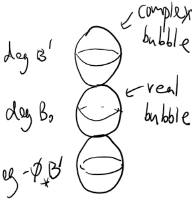
- real bubble is rigid $\Rightarrow$ #(nodal curves)= #(real bubble)$\cdot$#(complex bubble, with an additional constraint)
- complex bubble is rigid $\Rightarrow$ #(nodal curves)= #(complex bubble)$\cdot$#(real bubble, with an additional complex constraint)
  We need to determine the homology class of this constraint in $H_*(X - X^\phi)$.

Sketch of proof – splitting

We need to determine the homology class of the complex bubble in $H_*(X - X^\phi)$.



Sketch of proof – splitting



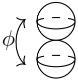
We need to determine the homology class of the complex bubble in $H_*(X - X^\phi)$.

Idea: use a finite group action $G \subset \text{Aut}(X, \omega, \phi)$ s.t.

$$H_2(X - X^\phi)^G \approx H_2(X).$$

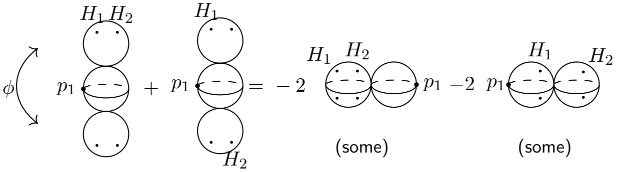
Take constraints $H_1, \ldots, H_l$ to be G-invariant $\implies$ The complex bubble is G-invariant, and thus determined.

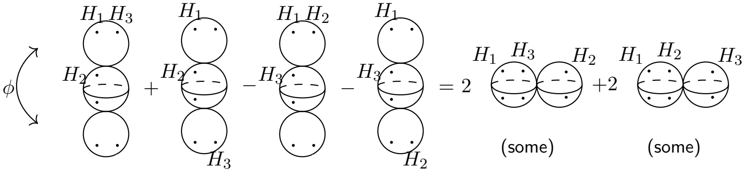
A remark on G

In the presence of such a group action, the invariants are defined even when sphere bubbling (curves of shape 

) appears — the group action cancels such things in pairs.

Real WDVV





+

splitting formulas

$\Downarrow$

WDVV-type relations for Welschinger invariants

Thank you!